

# Point-form quantum field theory

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## Abstract

We examine canonical quantization of relativistic field theories on the forward hyperboloid, a Lorentz-invariant surface of the form  $x_\mu x^\mu = \tau^2$ . This choice of quantization surface implies that all components of the 4-momentum operator are affected by interactions (if present), whereas rotation and boost generators remain interaction free – a feature characteristic of Dirac’s “point-form” of relativistic dynamics. Unlike previous attempts to quantize fields on space-time hyperboloids, we keep the usual plane-wave expansion of the field operators and consider evolution of the system generated by the 4-momentum operator. We verify that the Fock-space representations of the Poincaré generators for free scalar and spin-1/2 fields look the same as for equal-time quantization. Scattering is formulated for interacting fields in a covariant interaction picture and it is shown that the familiar perturbative expansion of the S-operator is recovered by our approach. An appendix analyzes special distributions, integrals over the forward hyperboloid, that are used repeatedly in the paper.

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## 1 Introduction

One of the main goals of formulating a relativistic many-body theory is to find a realization of the Poincaré algebra in terms of operators which act on a Fock

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space. From knowledge of the Poincaré generators it is then possible to calculate physical observables like the mass spectrum, scattering data, or probability distributions in different inertial frames. For systems of free particles the Fock-space representation of the Poincaré generators follows immediately from their representations on one-particle Hilbert spaces. For interacting many-body systems the situation is much more complicated. A close inspection of the Poincaré algebra reveals that interaction terms must appear in more than one Poincaré generator and are, in general, constrained by non-linear relations which are hard to satisfy. A possible solution to this problem is to start with a classical field theory, which is specified by a (Lorentz-)scalar Lagrangian density, and quantize it. The operators which generate Poincaré transformations then automatically satisfy the Poincaré algebra, even for interacting theories. According to Tomonaga [1] and Schwinger [2] (see also Ref. [3]) quantization can be carried out on an arbitrary space-like hypersurface of Minkowski space-time by imposing (generalized) canonical (anti)commutation relations on the field operators. Evolution of the system from one such hypersurface to the neighboring one may then be described in a Lorentz-invariant way by means of the, so called, “Tomonaga-Schwinger equation”.

We make use of this general framework and quantize field theories on the Lorentz-invariant space-time hyperboloid  $\Sigma : x_\mu x^\mu = \tau^2$ ,  $\tau$  arbitrary but fixed. For interacting theories the choice of the quantization surface is intimately connected with the interaction dependence of the Poincaré generators. In our case all components of the 4-momentum operator become interaction dependent, whereas the generators of Lorentz transformations stay free of interactions. This resembles Dirac’s point form of classical relativistic dynamics [4]. Therefore we speak of “point-form quantum field theory”(PFQFT). For the usual equal-time quantization, which corresponds to Dirac’s instant form, interaction terms appear in the generator of time translations, i.e.  $\hat{P}^0$ , and in the generators of Lorentz boosts  $\hat{K}_i$ ,  $i = 1, 2, 3$ . In either case, the interaction-free generators give rise to a subgroup of the Poincaré group, the so called, “kinematic subgroup” or “stability group”, which leaves the quantization surface invariant. Since there exist 3 additional continuous subgroups of the Poincaré group one may think of other preferred choices of quantization surfaces [5]. Actually, only field quantization at equal time  $t$ , equal light-cone time  $x_+ = t + x_3$ , and on space-time hyperboloids  $x_\mu x^\mu = \tau^2$  has been discussed in the literature. Equal-time quantization is common text-book knowledge. Field quantization on the light front is also well developed and has attracted interest in connection with hard hadronic processes and with the solution of the QCD bound-state problem [6,7].

But only a few old papers exist which are dedicated to PFQFT [8,9,10,11,12,13]. The reason is, of course, the curved nature of the quantization surface which poses some technical problems. Nevertheless, from a conceptual point of view PFQFT is rather attractive. The interaction-dependent, dynamical Poincaré

generators are components of a 4-vector and the interaction-free, kinematical Poincaré generators can be combined to a second-order tensor. This makes a manifestly Lorentz-covariant formulation of PFQFT feasible, as had already been noticed by Dirac [4].

The benefits of the point form have up till now only been exploited in relativistic quantum mechanics for the analysis of electromagnetic current operators [14,15,16] and the calculation of masses [17,18], electroweak properties [19,20] and strong decays of hadrons [21,22] within constituent quark models. Ref. [23] takes advantage of point form relativistic quantum mechanics to develop a Poincaré invariant coupled-channel formalism in which the interaction vertices are derived from quantum field theory. First applications of this formalism include the calculation of (axial) vector-meson masses and decays within the chiral constituent quark model [24,25]. This coupled-channel formalism can also be understood as a truncation scheme for quantum field theories which preserves Poincaré invariance. The additional approximation which enters is the assumption that the total velocity of the system is conserved at interaction vertices. In fact, quantum field theoretical ideas form the background for most applications of point form relativistic quantum mechanics. The starting point for the construction of particle-exchange potentials or current- and decay-operators which are used in relativistic quantum mechanics is usually a quantum field theory. It is thus important to put the operator formalism for field quantization on space-time hyperboloids on a solid footing.

Since we aim at a Fock-space representation of Lagrangian field theories we have to specify a Fock space. Fock spaces are infinite direct sums of tensor products of single-particle Hilbert spaces. Each single-particle space is a representation space for a unitary irreducible representation of the Poincaré group. General one-particle states, representing a particle with certain mass and spin, may thus be expressed as a superposition of eigenstates of a complete set of commuting self-adjoint operators that is constructed from the Poincaré generators. The basis most commonly used is the, so called, “Wigner basis” which consists of simultaneous eigenstates of the 3-momentum operator and an additional operator describing the spin orientation. This basis diagonalizes part of the generators of the kinematic subgroup for equal-time quantization, i.e. those of the Abelian subgroup of translations. However, the old papers on PFQFT made use of another basis, which was obtained by reducing the Poincaré group with respect to the Lorentz subgroup, the, so called, “Lorentz basis”. In this basis the Casimir operator of the homogeneous Lorentz group as well as the operators for the total angular momentum and one of its components are diagonalized simultaneously [10]. But the big disadvantage of such a basis is that the 4-momentum operator cannot easily be defined as a self-adjoint operator acting on square-integrable functions [26]. Since a primary goal is to show the equivalence of quantization on space-time hyperboloids and equal-time quantization for free and simple interacting field theories we will

stick with the usual basis of eigenstates of the (free) 3-momentum operator.

Often point-form is associated with evolution of the system in the parameter  $\tau$ , i.e. perpendicular to the hyperboloid on which field quantization takes place. This kind of evolution is generated by the dilatation operator and has been studied in the old papers on PFQFT [8,9,10,11,12]. At first sight it seems to be quite natural to consider evolution in  $\tau$  as soon as one introduces hyperbolic coordinates to parameterize the hyperboloid  $x_\mu x^\mu = \tau^2$ . But evolution in  $\tau$  also gives rise to some problems. If one is not dealing with a massless theory, the dilatation operator is  $\tau$ -dependent. With hyperbolic coordinates one needs at least 3 different coordinate patches to cover the whole Minkowski space-time [12]. It thus becomes rather cumbersome to follow the evolution of the system from the backward to the forward light cone as is, e.g., necessary if one wants to formulate scattering. Looking back at Dirac's seminal paper on Hamiltonian formulations of classical relativistic dynamics no reference is made to a particular choice of a time parameter. The different forms are only characterized by the space-like hypersurface of Minkowski space-time on which the initial conditions are posed and those Poincaré generators which do not generate the kinematic subgroup are denoted as "Hamiltonians" [4]. The Hamiltonians tell us how the dynamical variables of the system evolve under the corresponding Poincaré transformations. Their knowledge suffices to calculate the evolution of the system from the distant past to the far future in any inertial frame. The situation is quite the same for the operator approach to quantum field theories. In the case of PFQFT the Fock-space representation of the 4-momentum operator already contains all the information which is necessary for the calculation of the mass spectrum and the scattering matrix. It will thus be a primary task to show how the Fock-space representation of the 4-momentum operator looks if a spin-zero or spin-1/2 field is quantized on the forward hyperboloid. Since we will use the usual momentum state basis, differences with equal-time quantization are only to be expected for interacting fields.

Sec. 2 elucidates the problems encountered in previous attempts to formulate PFQFT for the simplest case of a real scalar field theory in 1+1 dimensional space-time. After having realized that these problems are mainly connected with the Lorentz basis and the evolution in  $\tau$ , we will switch to the usual Wigner basis and concentrate on the evolution of the system generated by the 4-momentum operator. The equivalence of equal- $\tau$  and equal-time quantization for free spin-0 and spin-1/2 fields is proved in Sec. 3. Thereby it turns out that all the necessary integrations over the forward hyperboloid can be carried out in Cartesian coordinates with the help of an appropriately defined distribution  $W(P, Q)$ . A manifestly covariant formulation of scattering is then developed in Sec. 4. It is shown that the perturbative expansion of the S-operator is equivalent to usual time-ordered perturbation theory. A summary of our findings and an outlook to further applications can be found in

Sec. 5. Finally the distribution  $W(P, Q)$  and its properties are discussed in some detail in App. A.

## 2 Historical attempts

To the best of our knowledge all the historical papers on PFQFT in Minkowski space-time did not go far beyond free fields [9,10,12]. The reasons can already be demonstrated for the simplest case of a real scalar field in 1+1 dimensional space-time. To see this we briefly recall the strategy of Refs. [9,12]. The starting point is the Lagrangian density, which for a free massive scalar field takes on the form

$$\mathcal{L}(t, x) = \frac{1}{2} \left[ (\partial_t \phi(t, x))^2 - (\partial_x \phi(t, x))^2 - m^2 \phi^2(t, x) \right]. \quad (1)$$

Since we want to quantize the theory on the hyperboloid  $t^2 - x^2 = \tau^2$  we go over to hyperbolic coordinates (with  $\tau = e^\alpha$ )

$$t = e^\alpha \cosh \beta, \quad x = e^\alpha \sinh \beta, \quad -\infty < \alpha, \beta < \infty. \quad (2)$$

With this change of coordinates our considerations are restricted to the forward light cone. The Lagrangian density expressed in terms of hyperbolic coordinates is

$$\mathcal{L}(\alpha, \beta) = \frac{1}{2} \left[ (\partial_\alpha \phi(\alpha, \beta))^2 - (\partial_\beta \phi(\alpha, \beta))^2 - e^{2\alpha} m^2 \phi^2(\alpha, \beta) \right]. \quad (3)$$

Via the action principle it gives rise to the Klein-Gordon equation in hyperbolic coordinates:

$$\left( \partial_\alpha^2 - \partial_\beta^2 + e^{2\alpha} m^2 \right) \phi(\alpha, \beta) = 0. \quad (4)$$

Since  $\beta$  parameterizes the hyperboloid it is natural to consider  $\alpha$  as time parameter and proceed analogous to canonical quantization at equal time (with  $x$  replaced by  $\beta$  and  $t$  by  $\alpha$ ). The canonical momentum conjugate to  $\phi(\alpha, \beta)$  is

$$\pi(\alpha, \beta) = \frac{\partial \mathcal{L}(\alpha, \beta)}{\partial (\partial_\alpha \phi(\alpha, \beta))}, \quad (5)$$

and the Hamiltonian (i.e. the Legendre transform of the Lagrangian) which generates “translations” in  $\alpha$  may be identified (in 1+1 dimensional space-

time) with the dilatation generator

$$D(\alpha) = \frac{1}{2} \int_{-\infty}^{\infty} \left[ (\partial_{\alpha}\phi(\alpha, \beta))^2 + (\partial_{\beta}\phi(\alpha, \beta))^2 + e^{2\alpha} m^2 \phi^2(\alpha, \beta) \right] d\beta. \quad (6)$$

Note that  $D(\alpha)$  is explicitly  $\alpha$ -dependent (i.e.  $\partial_{\alpha}D(\alpha) \neq 0$ ) and does not belong to the set of Poincaré generators.

According to the rules for canonical quantization the classical fields  $\phi(\alpha, \beta)$  and  $\pi(\alpha, \beta)$  have now to be replaced by corresponding operators  $\hat{\phi}(\alpha, \beta)$  and  $\hat{\pi}(\alpha, \beta)$ , respectively, which have to satisfy equal- $\alpha$  commutation relations

$$\begin{aligned} [\hat{\phi}(\alpha, \beta), \hat{\pi}(\alpha, \beta')] &= i\delta(\beta - \beta'), \\ [\hat{\phi}(\alpha, \beta), \hat{\phi}(\alpha, \beta')] &= [\hat{\pi}(\alpha, \beta), \hat{\pi}(\alpha, \beta')] = 0. \end{aligned} \quad (7)$$

Here we have used the Heisenberg representation. The Heisenberg equations of motion

$$\begin{aligned} \partial_{\alpha}\hat{\phi}(\alpha, \beta) &= -i [\hat{\phi}(\alpha, \beta), \hat{D}(\alpha)] = \hat{\pi}(\alpha, \beta), \\ \partial_{\alpha}\hat{\pi}(\alpha, \beta) &= -i [\hat{\pi}(\alpha, \beta), \hat{D}(\alpha)], \end{aligned} \quad (8)$$

follow immediately from the canonical commutation relations, Eqs. (7), with  $\hat{D}(\alpha)$  being the (quantized) dilatation operator. Equations (8) further imply that the field operator  $\hat{\phi}(\alpha, \beta)$  satisfies the original Klein-Gordon equation, Eq. (4).

The construction of the Fock space usually starts with a choice of basis states for the one-particle Hilbert space. To this aim the field operator is expanded in terms of solutions of the Klein-Gordon equation, Eq. (4), which are orthogonal under the  $\alpha$ -independent scalar product

$$(\phi, \psi)_{\alpha} = i \int_{-\infty}^{\infty} d\beta \left[ \phi^*(\alpha', \beta) \frac{\partial \psi(\alpha', \beta)}{\partial \alpha'} - \frac{\partial \phi^*(\alpha', \beta)}{\partial \alpha'} \psi(\alpha', \beta) \right]_{\alpha'=\alpha}. \quad (9)$$

Imitating equal-time quantization it is assumed that the  $\alpha$ - and  $\beta$ -dependence of these solutions factorizes and the  $\beta$ -dependent part is simply a plane wave. An appropriately normalized  $((\phi_{\lambda}, \phi_{\lambda'})_{\alpha} = \delta(\lambda - \lambda'))$  complete set of solutions, which meets these requirements, is given by

$$\phi_{\lambda}(\alpha, \beta) = -\frac{ie^{\frac{\pi}{2}\lambda}}{\sqrt{8}} H_{i\lambda}^{(2)}(me^{\alpha}) e^{i\lambda\beta} \quad \text{and}$$

$$\phi_\lambda^*(\alpha, \beta) = \frac{ie^{-\frac{\pi}{2}\lambda}}{\sqrt{8}} H_{i\lambda}^{(1)}(me^\alpha) e^{-i\lambda\beta}, \quad (10)$$

with  $-\infty < \lambda < \infty$  and  $H_{i\lambda}^{(\cdot)}$  denoting Hankel functions. The reason for taking Hankel functions for the  $\alpha$ -dependence is that they satisfy the same boundary conditions as in equal-time quantization, i.e.  $\phi_\lambda(\alpha, \beta)$  are solutions of the Klein-Gordon equation which travel forward in (ordinary) time  $t$ . Consequently, these solutions lead to the usual Feynman propagator [12]. Ref. [9], on the other hand, has taken solutions which travel forward in  $\alpha$ . The field quanta introduced in this way, however, do not coincide with the usual Poincaré invariant definition of a particle. The expansion of the field operator  $\hat{\phi}(\alpha, \beta)$  in terms of the functions  $\phi_\lambda$  reads

$$\hat{\phi}(\alpha, \beta) = \int_{-\infty}^{\infty} d\lambda \left[ \hat{b}_\lambda \phi_\lambda(\alpha, \beta) + \hat{b}_\lambda^\dagger \phi_\lambda^*(\alpha, \beta) \right], \quad (11)$$

with the ( $\alpha$ -independent) “Fourier coefficients”  $\hat{b}_\lambda$  and  $\hat{b}_\lambda^\dagger$  being given by

$$\hat{b}_\lambda = (\phi_\lambda, \hat{\phi})_\alpha, \quad \hat{b}_\lambda^\dagger = -(\phi_\lambda^*, \hat{\phi})_\alpha. \quad (12)$$

Equations (12) and the equal- $\alpha$  commutation relations, Eqs. (7), imply the harmonic-oscillator commutation relations

$$[\hat{b}_\lambda, \hat{b}_{\lambda'}^\dagger] = \delta(\lambda - \lambda'), \quad [\hat{b}_\lambda, \hat{b}_{\lambda'}] = [\hat{b}_\lambda^\dagger, \hat{b}_{\lambda'}^\dagger] = 0. \quad (13)$$

The operators  $\hat{b}_\lambda^\dagger$  and  $\hat{b}_\lambda$  can be interpreted as creation and annihilation operators of field quanta which are characterized by a real value  $\lambda$ . The physical interpretation of  $\lambda$  is that of an eigenvalue of  $\hat{K}$ , the generator of Lorentz boosts. In the  $\lambda$ -basis the operator  $\hat{K}$ , as calculated from the stress-energy tensor, becomes diagonal [27]. Its Fock-space representation is

$$\hat{K} = \int_{-\infty}^{\infty} d\lambda \lambda \hat{b}_\lambda^\dagger \hat{b}_\lambda. \quad (14)$$

This means in particular that  $|\lambda\rangle = \hat{b}_\lambda^\dagger |0\rangle$ , with  $|0\rangle$  denoting the vacuum state, is an eigenstate of  $\hat{K}$ , i.e.  $\hat{K}|\lambda\rangle = \lambda|\lambda\rangle$ .

Unlike the boost generator, the dilatation generator  $\hat{D}(\alpha)$  is, in general, not diagonalized by the boost eigenstates  $|\lambda\rangle$ . Even for the interaction-free case its Fock-space representation has a complicated structure (for brevity we have neglected the arguments  $me^\alpha$  of the Hankel functions):

$$\begin{aligned} \hat{D}(\alpha) = & \frac{\pi}{8} m^2 e^{2\alpha} \int_{-\infty}^{\infty} d\lambda \left\{ \left[ 2H_{i\lambda}^{(2)} H_{i\lambda}^{(1)} - H_{i\lambda-1}^{(2)} H_{i\lambda+1}^{(1)} - H_{i\lambda+1}^{(2)} H_{i\lambda-1}^{(1)} \right] \hat{b}_\lambda \hat{b}_\lambda^\dagger \right. \\ & \left. - 2 \left[ H_{i\lambda}^{(2)} H_{-i\lambda}^{(2)} + H_{1+i\lambda}^{(2)} H_{1-i\lambda}^{(2)} \right] \hat{b}_\lambda \hat{b}_\lambda + h.c. \right\} . \end{aligned} \quad (15)$$

Single particle states are therefore not eigenstates of the dilatation operator  $\hat{D}(\alpha)$ .

Our discussion has up till now been confined to the forward light cone. As has been shown in Ref. [12] this restriction can be overcome by analytic continuation of  $\phi_\lambda$  along appropriately chosen (complex) paths for  $\alpha$  and  $\beta$ . In this way  $\phi_\lambda$  is represented by 4 different functions, each belonging to one of the 4 wedges of Minkowski space-time. Evolution of the (quantized) fields can then be considered from surface to surface with the surfaces being hyperboloids in the forward and backward light cone and cones for  $x^2 < 0$ , respectively. This means that outside the light cone the hyperbolic coordinates  $\alpha$  and  $\beta$  essentially exchange their roles –  $\beta$  becomes the time parameter and  $\alpha$  labels the position on the cone. As a consequence  $\hat{D}(\alpha)$  cannot be used outside the light cone, but another generator for evolution in  $\beta$  has to be introduced. Altogether it does not seem to be very practical to study evolution of quantum field theories in hyperbolic coordinates. Exceptions are perhaps scale-invariant theories for which the mass  $m$  has to vanish. The limit  $m \rightarrow 0$ , however, does not easily follow from the formulas given above, but has to be considered separately [12].

Another problem with the kind of approach just outlined is connected with the  $\lambda$ -representation. This representation diagonalizes the boost generator  $\hat{K}$ , but it complicates matters for the momentum operator. To see the reason we first express the single-particle boost eigenstates  $|\lambda\rangle$  in terms of momentum eigenstates  $|p\rangle$ ,

$$|\lambda\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\omega_p} |p\rangle \langle p|\lambda\rangle , \quad (16)$$

with  $\omega_p = \sqrt{m^2 + p^2}$  and

$$\langle p|\lambda\rangle = (\phi_p, \phi_\lambda)_\alpha = \frac{1}{\sqrt{\pi}} \left( \frac{p + \omega_p}{m} \right)^{i\lambda} = \frac{1}{\sqrt{\pi}} e^{i\lambda\chi} . \quad (17)$$

For practical calculations it is often convenient to replace the momentum  $p$  by the variable  $\chi$ , which is defined via  $p = m \sinh \chi$  and  $\omega_p = m \cosh \chi$ .  $\phi_p$  are usual plane waves expressed in terms of hyperbolic coordinates

$$\phi_p(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} e^{-i(\omega_p t(\alpha, \beta) - p x(\alpha, \beta))} = \frac{1}{\sqrt{2\pi}} e^{-im \exp(\alpha) \cosh(\chi - \beta)} . \quad (18)$$



With the help of Eq. (17) we can now see that the spatial component of the momentum operator  $\hat{P}^1$  shifts  $\lambda$  by an imaginary quantity

$$\begin{aligned}
\langle \lambda' | \hat{P}^1 | \lambda \rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\omega_p} \int_{-\infty}^{\infty} \frac{dp'}{2\omega_{p'}} \langle \lambda' | p' \rangle \langle p' | \hat{P}^1 | p \rangle \langle p | \lambda \rangle \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\omega_p} \int_{-\infty}^{\infty} \frac{dp'}{2\omega_{p'}} \frac{1}{\sqrt{\pi}} \left( \frac{p' + \omega_{p'}}{m} \right)^{-i\lambda'} (p 2\omega_p \delta(p - p')) \frac{1}{\sqrt{\pi}} \left( \frac{p + \omega_p}{m} \right)^{i\lambda} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dp}{\omega_p} p \left( \frac{p + \omega_p}{m} \right)^{i(\lambda - \lambda')} = \frac{m}{2\pi} \int_{-\infty}^{\infty} d\chi \sinh(\chi) e^{i\chi(\lambda - \lambda')} \\
&= \frac{m}{4\pi} \int_{-\infty}^{\infty} d\chi \left( e^{i\chi(\lambda - \lambda' - i)} - e^{i\chi(\lambda - \lambda' + i)} \right) \\
&= \frac{m}{2} [\delta(\lambda - \lambda' - i) - \delta(\lambda - \lambda' + i)] .
\end{aligned} \tag{19}$$

For  $\hat{P}^0$  the minus sign between the two delta functions has to be replaced by a plus sign. The meaning of this result has been clarified in Ref. [26]. Since  $\hat{P}^\mu$  are unbounded operators they are only defined on a subspace of the one-particle Hilbert space  $\mathcal{L}^2((-\infty, \infty); dp/(2\omega_p))$ . By means of the transformation given in Eq. (17) this subspace goes over into a subspace of square integrable functions in  $\lambda$  that is characterized by the property that its elements are analytic in a strip  $|\text{Im } \lambda| < 1$ . The matrix elements  $\langle \lambda' | \hat{P}^\mu | \lambda \rangle$ ,  $\mu = 0, 1$ , therefore have to be understood as distributions acting on square-integrable functions in  $\lambda$  which are analytic in the strip  $|\text{Im } \lambda| < 1$ . For such test functions the action of  $\hat{P}^\mu$  is well defined:  $\hat{P}^\mu |f\rangle = |f'\rangle$  with  $f'(\lambda) = \langle \lambda | f' \rangle = m(f(\lambda + i) + (-1)^\mu f(\lambda - i))/2$ . The situation seems to be similar to the case where one studies unitary representations of noncompact groups and tries to diagonalize the operators which do not generate a compact subgroup [28]. Altogether, the definition of the translation generators as selfadjoint operators acting on square integrable functions of  $\lambda$  obviously needs special care and is, at least, not completely straightforward.

The generalization of the quantization procedure sketched above to (free) complex scalar and spin-1/2 fields in 3+1-dimensional Minkowski space-time has been worked out in Refs. [9,12] and in Ref. [10], respectively. In 3+1 dimensions the orthogonal set of basis functions  $\phi_\lambda(\alpha, \beta)$  (for scalar fields) has to be replaced by another set of functions which are labelled by 3 parameters and which now depend on 3 spatial coordinates (usually  $\beta$ ,  $\theta$ , and  $\varphi$ ). References [9,12] take  $\phi_{\lambda,z}$  with  $z$  being an arbitrary complex number, whereas Ref. [10] rather uses  $\phi_{\lambda,l,m}$  with integers  $l = 0, 1, 2, \dots$  and  $m = -l, -l + 1, \dots, l - 1, l$ . In both cases  $(1 + \lambda^2)$ , with  $0 \leq \lambda < \infty$ , have to be understood as eigenvalues of the Casimir operator  $\vec{K}^2 - \vec{L}^2$  of the homoge-

neous Lorentz group. Its unitary irreducible representations can therefore be characterized by  $\lambda$  and the parameters  $z$  or  $(l, m)$  label different orthogonal sets of basis vectors of the representation space. The problems due to hyperbolic coordinates and with the  $\lambda$ -representation of the 4-momentum operator are, of course, also present in the 3+1 dimensional case. These problems can be avoided in Euclidean field theories. For the Euclidean version of PFQFT the hyperbolic coordinates are replaced by spherical coordinates, which can be defined globally. Quantization is then done on a sphere and evolution is considered into radial direction. Since boosts are described by 4-dimensional rotations in the Euclidean theory, the boost eigenvalues  $\lambda$  assume integer values. The  $\lambda$ -representation of the 4-momentum operator contains still non-diagonal terms, but  $\hat{P}^\mu$  shifts  $\lambda$  only by  $\pm 1$ . A discussion of the Euclidean formulation for massless spin-0 and spin-1/2 fields in 2 and 4 dimensions can be found in Ref. [8]. The corresponding massive cases in 2 dimensions are considered in Ref. [11].

Nowadays the radial quantization procedure of Fubini et al. [8] is the standard method for quantization of 2-dimensional, conformally symmetric models for which the dilatation generator  $\hat{D}$  is a constant of motion [29]. But it does not seem to be very useful in applications to massive theories if one is not primarily interested in the spectrum of the dilatation generator, but rather in the spectrum of the mass operator. For this purpose it is more convenient to have a simple representation of the 4-momentum operator  $\hat{P}^\mu$  and pay less attention to the dilatation generator  $\hat{D}$ . Also, a Lorentz-invariant formulation of scattering does not necessarily depend on evolution generated by  $\hat{D}$ , but can as well be achieved in terms of the 4-momentum operator. In the following we will thus make use of the usual basis of momentum eigenstates and investigate the evolution of the system that is generated by  $\hat{P}^\mu$ .

### 3 Quantization of free fields

In this section we will show for free field theories that the usual (momentum) Fock-space representation of the Poincaré generators, that is well known from equal-time quantization, follows also from quantization on the hyperboloid  $x_\mu x^\mu = \tau^2$ . We will start with the case of a complex scalar field.

### 3.1 Spin-0 fields

The Lagrangian density for a free scalar field in 3+1 dimensional Minkowski space-time is

$$\mathcal{L}_{\text{free}}(x) = \left[ (\partial_\mu \phi^*(x)) (\partial^\mu \phi(x)) - m^2 \phi^*(x) \phi(x) \right], \quad (20)$$

where  $x$  denotes the contravariant 4-vector  $(x^\mu) = (t, \vec{x})$ . The equation of motion which follows from  $\mathcal{L}_{\text{free}}(x)$  is the Klein-Gordon equation

$$\left( \partial_\mu \partial^\mu + m^2 \right) \phi(x) = 0. \quad (21)$$

An important statement which can be made for arbitrary solutions  $\psi(x)$  and  $\chi(x)$  of the Klein-Gordon equation, Eq. (21), is the following:

The scalar product

$$\begin{aligned} (\psi, \chi)_\sigma &:= i \int_\sigma d\sigma^\mu(x) [\psi^*(x) \partial_\mu \chi(x) - \chi(x) \partial_\mu \psi^*(x)] \\ &= i \int_\sigma d\sigma^\mu(x) \left[ \psi^*(x) \overleftrightarrow{\partial}_\mu \chi(x) \right], \end{aligned} \quad (22)$$

with  $\sigma$  denoting a space-like hypersurface of Minkowski space-time, does not depend on  $\sigma$ .

The general proof of this statement can be found in Ref. [3]. For a hyperplane of fixed time,

$$\sigma_t : x^0 = t \quad \implies \quad d\sigma^\mu = d^3x g^{0\mu}, \quad (23)$$

Eq. (22) represents nothing else than the well known fact that the scalar product

$$(\psi, \chi)_{\sigma_t} = i \int_{\mathbb{R}^3} d^3x \left[ \psi^*(t', \vec{x}) \overleftrightarrow{\partial}_{t'} \chi(t', \vec{x}) \right]_{t'=t} \quad (24)$$

does not depend on  $t$ . Now we will demonstrate that the scalar product  $(\psi, \chi)_\sigma$  for

$$\sigma_\tau : x_\mu x^\mu = \tau^2 \quad \implies \quad d\sigma^\mu(x) = 2 d^4x \delta(x \cdot x - \tau^2) \theta(x^0) x^\mu, \quad (25)$$

is independent of the chosen hyperboloid, i.e. it does not depend on  $\tau$ . Since every solution  $\phi(x)$  of the Klein-Gordon equation (21) may be decomposed into plane waves

$$\phi(x) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{3/2} 2\omega_{\vec{p}}} \tilde{\phi}(\vec{p}) e^{-ix \cdot p}, \quad (26)$$

with  $p = (p^\mu) = (\omega_{\vec{p}}, \vec{p})$  and  $\omega_p = \sqrt{m^2 + \vec{p}^2}$ , the scalar product  $(\psi, \chi)_\sigma$  can be written as

$$\begin{aligned} (\psi, \chi)_{\sigma_\tau} &= i \int_{\mathbb{R}^4} 2 d^4x \delta(x \cdot x - \tau^2) \theta(x^0) x^\mu \left[ \psi^*(x) \overleftrightarrow{\partial}_\mu \chi(x) \right] \\ &= \int_{\mathbb{R}^3} \frac{d^3p'}{(2\pi)^{3/2} 2\omega_{\vec{p}'}} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{3/2} 2\omega_{\vec{p}}} \tilde{\psi}^*(\vec{p}') \tilde{\chi}(\vec{p}) \\ &\quad \times \int_{\mathbb{R}^4} 2 d^4x \delta(x \cdot x - \tau^2) \theta(x^0) x^\mu (p' + p)_\mu e^{ix \cdot (p' - p)}. \end{aligned} \quad (27)$$

The  $x$ -integral is now nothing else than a Lorentz invariant distribution  $W(P, Q)$ , with  $P = (p' + p)$  and  $Q = (p' - p)$ , which can be easily calculated in an appropriate frame. For its calculation and its properties we refer to App. A. By means of Eq. (A.7) we end up with

$$\begin{aligned} (\psi, \chi)_{\sigma_\tau} &= \int_{\mathbb{R}^3} \frac{d^3p'}{(2\pi)^{3/2} 2\omega_{\vec{p}'}} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{3/2} 2\omega_{\vec{p}}} \tilde{\psi}^*(\vec{p}') \tilde{\chi}(\vec{p}) W(p' + p, p' - p) \\ &= \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_{\vec{p}}} \tilde{\psi}^*(\vec{p}) \tilde{\chi}(\vec{p}), \end{aligned} \quad (28)$$

which does not depend on  $\tau$ . Eq. (28) implies, in particular, that usual plane waves ( $\tilde{\psi}(\vec{p}) \propto \delta^3(\vec{p} - \vec{p}')$  and  $\tilde{\chi}(\vec{p}) \propto \delta^3(\vec{p} - \vec{p}')$ ) are orthogonal on the hyperboloid  $\sigma_\tau$ .

In order to quantize the scalar field theory on the hyperboloid  $\sigma_\tau$  we demand the following (Lorentz-invariant) quantization conditions for the field operator  $\hat{\phi}(x)$ :

$$x^\mu [\hat{\phi}(y), \partial_\mu \hat{\phi}^\dagger(x)]_{x^2=y^2=\tau^2} = i x^0 \delta^3(\vec{y} - \vec{x}), \quad (29)$$

$$[\hat{\phi}(y), \hat{\phi}(x)]_{x^2=y^2=\tau^2} = [\hat{\phi}^\dagger(y), \hat{\phi}^\dagger(x)]_{x^2=y^2=\tau^2} = 0. \quad (30)$$

These quantization conditions generalize the 1+1 dimensional case, Eqs. (7), without making reference to a particular choice of a time parameter. Noting

that  $\partial/\partial\tau = (x \cdot x)^{-1/2} x^\mu \partial_\mu$  and going over to Cartesian coordinates, these quantization conditions are seen to agree with those of Ref. [10]. They can as well be considered as an unintegrated version of Schwinger's covariant quantization conditions adapted to the hyperboloid [2]. The next step is to expand the field operator  $\hat{\phi}(x)$  in terms of a complete set of solutions of the Klein-Gordon equation, which are orthogonal under the invariant scalar product defined by Eqs. (22) and (25). As we have seen before, we are allowed to take usual plane waves and write ( $p^0 = \omega_{\vec{p}} = \sqrt{m^2 + \vec{p}^2}$ )

$$\hat{\phi}(x) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{3/2} 2\omega_{\vec{p}}} \left[ \hat{a}(\vec{p}) e^{-ix \cdot p} + \hat{b}^\dagger(\vec{p}) e^{ix \cdot p} \right], \quad (31)$$

with the  $\tau$  independent “Fourier coefficients” given by

$$\begin{aligned} \hat{a}(\vec{p}) &= (\phi_{\vec{p}}, \hat{\phi})_{\sigma_\tau}, & \hat{a}^\dagger(\vec{p}) &= -(\phi_{\vec{p}}^*, \hat{\phi}^\dagger)_{\sigma_\tau}, \\ \hat{b}(\vec{p}) &= (\phi_{\vec{p}}, \hat{\phi}^\dagger)_{\sigma_\tau}, & \hat{b}^\dagger(\vec{p}) &= -(\phi_{\vec{p}}^*, \hat{\phi})_{\sigma_\tau}, \end{aligned} \quad (32)$$

and  $\phi_{\vec{p}}(x) = \exp(-ix \cdot p)/(2\pi)^{3/2}$ . These relations and the field commutators, Eqs. (29) and (30), imply the harmonic-oscillator commutation relations

$$[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p}')] = [\hat{b}(\vec{p}), \hat{b}^\dagger(\vec{p}')] = 2\omega_{\vec{p}} \delta^3(\vec{p} - \vec{p}'). \quad (33)$$

All the other commutators vanish. In deriving these commutation relations one encounters two integrations over space-time hyperboloids. One of these integrations is done by means of the delta function resulting from the field commutators, the remaining one is of the form  $W(P, Q)$  or  $W(Q, P)$  and can thus also be done easily (cf. App. A). Due to their commutation relations the operators  $\hat{a}(\vec{p})$ ,  $\hat{b}(\vec{p})$ ,  $\hat{a}^\dagger(\vec{p})$ , and  $\hat{b}^\dagger(\vec{p})$  may be interpreted as lowering or raising operators which annihilate or create field quanta characterized by a continuous label  $\vec{p}$ . As a result of the  $U(1)$ -symmetry of our scalar theory we expect the existence of a conserved charge operator

$$\hat{Q} = i \int_{\mathbb{R}^4} d^4x \delta(x \cdot x - \tau^2) \theta(x^0) x^\mu : \hat{\phi}^\dagger(x) \overleftrightarrow{\partial}_\mu \hat{\phi}(x) :, \quad (34)$$

where “ $: \dots :$ ” denotes usual normal ordering. Inserting the field expansion, Eq. (31), we get again  $x$ -integrals of the form  $W(P, Q)$ , or  $W(Q, P)$  which can be used to do one of the  $p$ -integrations. Finally one ends up with the usual form for the charge operator

$$\hat{Q} = \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_{\vec{p}}} \left[ \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) - \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) \right]. \quad (35)$$

This suggests that  $\hat{a}^\dagger(\vec{p})$  can be considered as a creation operator for particles of charge +1 and  $\hat{b}^\dagger(\vec{p})$  as a creation operator of antiparticles with charge -1.

Next we will express the generators of space-time translations in terms of annihilation and creation operators. The starting point is the energy-momentum tensor (for brevity we neglect the argument of  $\hat{\phi}(x)$ ):

$$\hat{T}^{\mu\nu}(x) =: \partial^\mu \hat{\phi}^\dagger \partial^\nu \hat{\phi} + \partial^\nu \hat{\phi}^\dagger \partial^\mu \hat{\phi} - g^{\mu\nu} \mathcal{L}(\hat{\phi}, \hat{\phi}^\dagger, \partial_\alpha \hat{\phi}, \partial_\alpha \hat{\phi}^\dagger) : . \quad (36)$$

For later purposes we note that this expression is also valid for interacting theories as long as the interaction terms do not contain derivatives of the fields. The 4-momentum operator, which generates translations of the field operator in the sense that  $\partial^\mu \hat{\phi}(x) = i[\hat{P}^\mu, \hat{\phi}(x)]$ , is then given by

$$\hat{P}^\mu = \int_{\sigma_\tau} d\sigma_\nu \hat{T}^{\nu\mu}(x) = \int_{\mathbb{R}^4} 2 d^4x \delta(x^2 - \tau^2) \theta(x^0) x_\nu \hat{T}^{\nu\mu}(x) . \quad (37)$$

Taking the Lagrangian density, Eq. (20), and inserting the field expansion, Eq. (31), into  $\hat{T}^{\mu\nu}(x)$  yields

$$\begin{aligned} \hat{P}_{\text{free}}^\mu = & \frac{1}{(2\pi)^3} \int_{\mathbb{R}^4} 2 d^4x \delta(x^2 - \tau^2) \theta(x^0) \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_{\vec{p}}} \int_{\mathbb{R}^3} \frac{d^3p'}{2\omega_{\vec{p}'}} \\ & \left\{ \left[ p'^\mu x \cdot p + p' \cdot x p^\mu - x^\mu (p \cdot p' - m^2) \right] \right. \\ & \times \left( e^{i(p'-p) \cdot x} \hat{a}^\dagger(\vec{p}') \hat{a}(\vec{p}) + e^{-i(p'-p) \cdot x} \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}') \right) \\ & - \left[ p'^\mu x \cdot p + p' \cdot x p^\mu - x^\mu (p \cdot p' + m^2) \right] \\ & \times \left( e^{i(p'+p) \cdot x} \hat{a}^\dagger(\vec{p}') \hat{b}^\dagger(\vec{p}) + e^{-i(p'+p) \cdot x} \hat{a}(\vec{p}) \hat{b}(\vec{p}') \right) \left. \right\} . \quad (38) \end{aligned}$$

Introducing new momenta  $P = p + p'$  and  $Q = p - p'$  we may rewrite the square brackets as  $[P^\mu x \cdot P - Q^\mu x \cdot Q + x^\mu Q \cdot Q]/2$  and  $[P^\mu x \cdot P - Q^\mu x \cdot Q - x^\mu P \cdot P]/2$ , respectively. Interchanging integrations, Eq. (38) becomes

$$\begin{aligned} \hat{P}_{\text{free}}^\mu = & \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_{\vec{p}}} \int_{\mathbb{R}^3} \frac{d^3p'}{2\omega_{\vec{p}'}} \\ & \times [P^\mu W(P, Q) - Q^\mu Q^\nu W_\nu(P, Q) + Q \cdot Q W^\mu(P, Q)] \hat{a}^\dagger(\vec{p}') \hat{a}(\vec{p}) \\ & + [P^\mu W(P, -Q) - Q^\mu Q^\nu W_\nu(P, -Q) + Q \cdot Q W^\mu(P, -Q)] \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}') \\ & - [P^\mu P^\nu W_\nu(Q, P) - Q^\mu W(Q, P) - P \cdot P W^\mu(Q, P)] \hat{a}^\dagger(\vec{p}') \hat{b}^\dagger(\vec{p}) \\ & - [P^\mu P^\nu W_\nu(Q, -P) - Q^\mu W(Q, -P) - P \cdot P W^\mu(Q, -P)] \hat{a}(\vec{p}) \hat{b}(\vec{p}') , \quad (39) \end{aligned}$$

with the  $x$ -integrals denoted as in App. A. With the results of App. A it is easily seen that only the  $P^\mu W(P, Q)$  terms survive and the remaining terms cancel each other. With the explicit expression for  $W(P, Q)$ , as given in Eq. (A.7), we finally recover the usual Fock-space representation of the 4-momentum operator

$$\begin{aligned}
\hat{P}_{\text{free}}^\mu &= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{d^3 p}{2\omega_{\vec{p}}} \int_{\mathbb{R}^3} \frac{d^3 p'}{2\omega_{\vec{p}'}} P^\mu \left\{ W(P, Q) \hat{a}^\dagger(\vec{p}') \hat{a}(\vec{p}) + W(P, -Q) \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}') \right\} \\
&= \frac{1}{2} \int_{\mathbb{R}^3} \frac{d^3 p}{2\omega_{\vec{p}}} \int_{\mathbb{R}^3} \frac{d^3 p'}{2\omega_{\vec{p}'}} P^\mu P^0 \delta^3(\vec{Q}) \left\{ \hat{a}^\dagger(\vec{p}') \hat{a}(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}') \right\} \\
&= \int_{\mathbb{R}^3} \frac{d^3 p}{2\omega_{\vec{p}}} p^\mu \left\{ \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) \right\} .
\end{aligned} \tag{40}$$

From this form of the 4-momentum operator and the commutation relations, Eqs. (29), we conclude that the field quanta created by  $\hat{a}^\dagger(\vec{p})$  and  $\hat{b}^\dagger(\vec{p})$  are eigenstates of the 4-momentum operator with eigenvalues  $p^\mu$ . The corresponding calculation for the boost and rotation generators is somewhat more tedious, but leads also to the well known result. This proves the equivalence of equal-time quantization and quantization on the hyperboloid  $x_\mu x^\mu = \tau^2$  for the case of free scalar fields.

### 3.2 Spin-1/2 fields

In the case of a free spin-1/2 field one can proceed in an analogous way. The Lagrangian density

$$\mathcal{L}_{\text{free}}(x) = \bar{\psi}(x) (i\gamma^\mu \partial_\mu - m) \psi(x) \tag{41}$$

leads to the Dirac equation

$$(i\gamma^\mu \partial_\mu - m) \psi(x) = 0. \tag{42}$$

For arbitrary solutions  $\psi(x)$  and  $\phi(x)$  of the Dirac equation, Eq. (42), the invariant scalar product on the hyperboloid reads [3]

$$\begin{aligned}
(\psi, \phi)_{\sigma_\tau} &= \int_{\sigma_\tau} d\sigma^\mu(x) \left[ \bar{\psi}(x) \gamma_\mu \phi(x) \right] \\
&= \int_{\mathbb{R}^4} 2 d^4 x \delta(x^2 - \tau^2) \theta(x^0) x^\mu \left[ \bar{\psi}(x) \gamma_\mu \phi(x) \right] .
\end{aligned} \tag{43}$$

A set of appropriately normalized solutions, which are orthogonal with respect to this scalar product, is given by

$$\psi_{\lambda, \vec{p}}^{(+)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-ip \cdot x} u_{\lambda}(\vec{p}), \quad \psi_{\lambda, \vec{p}}^{(-)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{ip \cdot x} v_{\lambda}(\vec{p}), \quad \lambda = \pm \frac{1}{2}, \quad (44)$$

with

$$u_{\lambda}(\vec{p}) = \sqrt{\omega_{\vec{p}} + m} \begin{pmatrix} \chi_{\lambda} \\ \frac{\vec{\sigma} \cdot \vec{p}}{\omega_{\vec{p}} + m} \chi_{\lambda} \end{pmatrix}, \quad v_{\lambda}(\vec{p}) = -\sqrt{\omega_{\vec{p}} + m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{\omega_{\vec{p}} + m} \hat{\epsilon} \chi_{\lambda} \\ \hat{\epsilon} \chi_{\lambda} \end{pmatrix}, \quad (45)$$

and

$$\chi_{\lambda} = \frac{1}{2} \begin{pmatrix} (1 + 2\lambda) \\ (1 - 2\lambda) \end{pmatrix}, \quad \hat{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (46)$$

The 4-spinors are normalized such that

$$\bar{u}_{\lambda}(\vec{p}) \gamma^{\mu} u_{\lambda'}(\vec{p}) = \bar{v}_{\lambda}(\vec{p}) \gamma^{\mu} v_{\lambda'}(\vec{p}) = 2p^{\mu} \delta_{\lambda\lambda'}. \quad (47)$$

Quantization of the spin-1/2 field  $\hat{\psi}(x)$  on the hyperboloid  $\sigma_{\tau}$  may be accomplished by demanding (Lorentz-invariant) anticommutation relations of the form:

$$x^{\mu} \left\{ \hat{\psi}_a(y), [\hat{\bar{\psi}}(x) \gamma_{\mu}]_b \right\}_{x^2=y^2=\tau^2} = x^0 \delta_{ab} \delta^3(\vec{x} - \vec{y}), \quad (48)$$

$$\left\{ \hat{\psi}_a(y), \hat{\psi}_b(x) \right\}_{x^2=y^2=\tau^2} = \left\{ \hat{\bar{\psi}}_a(y), \hat{\bar{\psi}}_b(x) \right\}_{x^2=y^2=\tau^2} = 0. \quad (49)$$

The subscripts “ $a$ ” and “ $b$ ” label the components of the 4-spinors  $\hat{\psi}$  and  $\hat{\bar{\psi}}$ . As in the scalar case these quantization conditions are consistent with those given in Refs. [2,10]. Creation and annihilation operators of field quanta with a particular momentum  $\vec{p}$  are introduced by expanding the field operator  $\hat{\psi}$  in terms of the plane waves given in Eq. (44):

$$\hat{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=\pm 1/2} \int_{\mathbb{R}^3} \frac{d^3 p}{2\omega_{\vec{p}}} \left( \hat{c}_{\lambda}(\vec{p}) u_{\lambda}(\vec{p}) e^{-ip \cdot x} + \hat{d}_{\lambda}^{\dagger}(\vec{p}) v_{\lambda}(\vec{p}) e^{ip \cdot x} \right). \quad (50)$$



The creation and annihilation operators are recovered by means of the invariant scalar product, Eq. (43),

$$\begin{aligned}\hat{c}_\lambda(\vec{p}) &= \left(\psi_{\lambda,\vec{p}}^{(+)}, \hat{\psi}\right)_{\sigma_\tau}, & \hat{c}_\lambda^\dagger(\vec{p}) &= \left(\hat{\psi}, \psi_{\lambda,\vec{p}}^{(+)}\right)_{\sigma_\tau}, \\ \hat{d}_\lambda(\vec{p}) &= \left(\hat{\psi}, \psi_{\lambda,\vec{p}}^{(-)}\right)_{\sigma_\tau}, & \hat{d}_\lambda^\dagger(\vec{p}) &= \left(\psi_{\lambda,\vec{p}}^{(-)}, \hat{\psi}\right)_{\sigma_\tau}.\end{aligned}\quad (51)$$

These relations are verified with the help of Eqs. (A.10) – (A.12), the spinor normalization condition, Eq. (47), and the fact that  $\bar{u}_\lambda(\vec{p})\gamma^\mu(p-p')_\mu u_{\lambda'}(\vec{p}') = \bar{u}_\lambda(\vec{p})\gamma^\mu(p+p')_\mu v_{\lambda'}(\vec{p}') = 0$ . The anticommutation relations for creation and annihilation operators follow from Eqs. (51) and the anticommutation relations for the field operators, Eqs. (48) and (49):

$$\left\{\hat{c}_\lambda(\vec{p}), \hat{c}_{\lambda'}^\dagger(\vec{p}')\right\} = \left\{\hat{d}_\lambda(\vec{p}), \hat{d}_{\lambda'}^\dagger(\vec{p}')\right\} = 2\omega_{\vec{p}} \delta_{\lambda\lambda'} \delta^3(\vec{p} - \vec{p}'). \quad (52)$$

All the other anticommutators vanish. For illustration we will in the following prove the anticommutation relation of  $\hat{c}_\lambda(\vec{p})$  with  $\hat{c}_{\lambda'}^\dagger(\vec{p}')$ :

$$\begin{aligned}\left\{\hat{c}_\lambda(\vec{p}), \hat{c}_{\lambda'}^\dagger(\vec{p}')\right\} &= \left\{\left(\psi_{\lambda,\vec{p}}^{(+)}, \hat{\psi}\right)_{\sigma_\tau}, \left(\hat{\psi}, \psi_{\lambda',\vec{p}'}^{(+)}\right)_{\sigma_\tau}\right\} \\ &= \int_{\mathbb{R}^4} 2d^4x \delta(x^2 - \tau^2) \theta(x^0) \int_{\mathbb{R}^4} 2d^4x' \delta(x'^2 - \tau^2) \theta(x'^0) \\ &\quad \times x^\mu x'^\nu \left\{\left[\bar{\psi}_{\lambda,\vec{p}}^{(+)}(x)\gamma_\mu\right]_a \left[\hat{\psi}(x)\right]_a, \left[\hat{\bar{\psi}}(x')\gamma_\nu\right]_b \left[\psi_{\lambda',\vec{p}'}^{(+)}(x')\right]_b\right\} \\ &= \frac{4}{(2\pi)^3} \int_{\mathbb{R}^4} d^4x \delta(x^2 - \tau^2) \theta(x^0) \int_{\mathbb{R}^4} d^4x' \delta(x'^2 - \tau^2) \theta(x'^0) \\ &\quad \times e^{ip\cdot x} e^{-ip'\cdot x'} x^\mu \left[\bar{u}_\lambda(\vec{p})\gamma_\mu\right]_a \left[u_{\lambda'}(\vec{p}')\right]_b x'^\nu \underbrace{\left\{\left[\hat{\psi}(x)\right]_a, \left[\hat{\bar{\psi}}(x')\gamma_\nu\right]_b\right\}}_{=x^0\delta_{ab}\delta^3(\vec{x}-\vec{x}')} \\ &= \frac{1}{(2\pi)^3} \left[\bar{u}_\lambda(\vec{p})\gamma_\mu u_{\lambda'}(\vec{p}')\right] \int_{\mathbb{R}^4} d^4x 2\delta(x^2 - \tau^2) \theta(x^0) x^\mu e^{-i(p'-p)\cdot x} \\ &= \frac{1}{(2\pi)^3} \left[\bar{u}_\lambda(\vec{p})\gamma_\mu u_{\lambda'}(\vec{p}')\right] W^\mu(p' + p, p' - p) \\ &= \underbrace{\left[\bar{u}_\lambda(\vec{p})\gamma_\mu u_{\lambda'}(\vec{p}')\right]}_{=2p_\mu \delta_{\lambda\lambda'}} \frac{2p^\mu}{4p\cdot p} 2\omega_{\vec{p}} \delta^3(\vec{p} - \vec{p}') \\ &= 2\omega_{\vec{p}} \delta_{\lambda\lambda'} \delta^3(\vec{p} - \vec{p}').\end{aligned}\quad (53)$$

Here we have used  $\bar{u}_\lambda(\vec{p})\gamma^\mu(p-p')_\mu u_{\lambda'}(\vec{p}') = 0$ , as well as Eqs. (A.7) and (A.9).

As in the scalar case we have a conserved charge operator

$$\hat{Q} = \int_{\mathbb{R}^4} 2 d^4x \delta(x \cdot x - \tau^2) \theta(x^0) x^\mu : \hat{\psi} \gamma_\mu \hat{\psi}(x) : . \quad (54)$$

After insertion of the field expansion, Eq. (50), into Eq.(54), one of the 3-dimensional momentum integrations can be performed by means of the delta function coming from the integration over the hyperboloid. The resulting Fock-space representation of the charge operator

$$\hat{Q} = \sum_{\lambda=\pm 1/2} \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_{\vec{p}}} \left[ \hat{c}_\lambda^\dagger(\vec{p}) \hat{c}_\lambda(\vec{p}) - \hat{d}_\lambda^\dagger(\vec{p}) \hat{d}_\lambda(\vec{p}) \right] \quad (55)$$

suggests that  $\hat{c}^\dagger(\vec{p})$  be considered as creation operators of particles with charge +1 and  $\hat{d}^\dagger(\vec{p})$  as creation operators of antiparticles with charge -1.

An appropriate version of the energy momentum tensor for the free spin-1/2 field is:

$$\hat{T}_{\text{free}}^{\mu\nu}(x) := \frac{i}{2} : \hat{\psi}(x) \gamma^\mu \overleftrightarrow{\partial}^\nu \hat{\psi}(x) : . \quad (56)$$

With this expression for  $\hat{T}^{\mu\nu}(x)$  the Fock-space representation of the 4-momentum operator becomes (cf. Eq. (37)):

$$\begin{aligned} \hat{P}_{\text{free}}^\mu &= \frac{i}{(2\pi)^3} \sum_{\lambda, \lambda'=\pm 1/2} \int_{\mathbb{R}^4} d^4x \delta(x^2 - \tau^2) \theta(x^0) \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_{\vec{p}}} \int_{\mathbb{R}^3} \frac{d^3p'}{2\omega_{\vec{p}'}} \\ &\quad \times : \left( \hat{c}_{\lambda'}^\dagger(\vec{p}') \bar{u}_\lambda(\vec{p}') e^{ip' \cdot x} + \hat{d}_\lambda(\vec{p}') \bar{v}_\lambda(\vec{p}') e^{-ip' \cdot x} \right) x_\nu \gamma^\nu \\ &\quad \times \overleftrightarrow{\partial}^\mu \left( \hat{c}_\lambda(\vec{p}) u_\lambda(\vec{p}) e^{-ip \cdot x} + \hat{d}_\lambda^\dagger(\vec{p}) v_\lambda(\vec{p}) e^{ip \cdot x} \right) : \\ &= \frac{1}{(2\pi)^3} \sum_{\lambda, \lambda'=\pm 1/2} \int_{\mathbb{R}^4} d^4x \delta(x^2 - \tau^2) \theta(x^0) \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_{\vec{p}}} \int_{\mathbb{R}^3} \frac{d^3p'}{2\omega_{\vec{p}'}} \\ &\quad \times : \left\{ (p+p')^\mu \left( e^{ix \cdot (p'-p)} \bar{u}_{\lambda'}(\vec{p}') x_\nu \gamma^\nu u_\lambda(\vec{p}) \hat{c}_{\lambda'}^\dagger(\vec{p}') \hat{c}_\lambda(\vec{p}) \right. \right. \\ &\quad \left. \left. - e^{-ix \cdot (p'-p)} \bar{v}_{\lambda'}(\vec{p}') x_\nu \gamma^\nu v_\lambda(\vec{p}) \hat{d}_{\lambda'}(\vec{p}') \hat{d}_\lambda^\dagger(\vec{p}) \right) \right. \\ &\quad \left. + (p-p')^\mu \left( e^{-ix \cdot (p'+p)} \bar{v}_{\lambda'}(\vec{p}') x_\nu \gamma^\nu u_\lambda(\vec{p}) \hat{d}_{\lambda'}(\vec{p}') \hat{c}_\lambda(\vec{p}) \right. \right. \\ &\quad \left. \left. - e^{ix \cdot (p'+p)} \bar{u}_{\lambda'}(\vec{p}') x_\nu \gamma^\nu v_\lambda(\vec{p}) \hat{c}_{\lambda'}^\dagger(\vec{p}') \hat{d}_\lambda^\dagger(\vec{p}) \right) \right\} : \\ &= \frac{1}{2(2\pi)^3} \sum_{\lambda, \lambda'=\pm 1/2} \int_{\mathbb{R}^3} \frac{d^3p}{2\omega_{\vec{p}}} \int_{\mathbb{R}^3} \frac{d^3p'}{2\omega_{\vec{p}'}} \\ &\quad \times : \left\{ P^\mu W_\nu(P, -Q) \bar{u}_{\lambda'}(\vec{p}') \gamma^\nu u_\lambda(\vec{p}) \hat{c}_{\lambda'}^\dagger(\vec{p}') \hat{c}_\lambda(\vec{p}) \right. \end{aligned}$$

$$\begin{aligned}
& -P^\mu W_\nu(P, Q) \bar{v}_{\lambda'}(\vec{p}') \gamma^\nu v_\lambda(\vec{p}) \hat{d}_{\lambda'}^\dagger(\vec{p}') \hat{d}_\lambda^\dagger(\vec{p}) \\
& + Q^\mu W_\nu(Q, P) \bar{v}_{\lambda'}(\vec{p}') \gamma^\nu u_\lambda(\vec{p}) \hat{d}_{\lambda'}^\dagger(\vec{p}') \hat{c}_\lambda(\vec{p}) \\
& - Q^\mu W_\nu(Q, -P) \bar{u}_{\lambda'}(\vec{p}') \gamma^\nu v_\lambda(\vec{p}) \hat{c}_{\lambda'}^\dagger(\vec{p}') \hat{d}_\lambda^\dagger(\vec{p}) \} : \\
& = \frac{1}{2} \sum_{\lambda, \lambda' = \pm 1/2} \int_{\mathbb{R}^3} \frac{d^3 p}{2\omega_{\vec{p}}} \int_{\mathbb{R}^3} \frac{d^3 p'}{2\omega_{\vec{p}'}} 2\omega_{\vec{p}'} \delta^3(\vec{p} - \vec{p}') \frac{P^\mu P_\nu}{P \cdot P} \\
& \quad \times \left\{ \bar{u}_{\lambda'}(\vec{p}') \gamma^\nu u_\lambda(\vec{p}) \hat{c}_{\lambda'}^\dagger(\vec{p}') \hat{c}_\lambda(\vec{p}) + \bar{v}_{\lambda'}(\vec{p}') \gamma^\nu v_\lambda(\vec{p}) \hat{d}_\lambda^\dagger(\vec{p}) \hat{d}_{\lambda'}^\dagger(\vec{p}') \right\} \\
& = \sum_{\lambda = \pm 1/2} \int_{\mathbb{R}^3} \frac{d^3 p}{2\omega_{\vec{p}}} p^\mu \left\{ \hat{c}_\lambda^\dagger(\vec{p}) \hat{c}_\lambda(\vec{p}) + \hat{d}_\lambda^\dagger(\vec{p}) \hat{d}_\lambda(\vec{p}) \right\} . \tag{57}
\end{aligned}$$

Here we have again used the the fact that  $\bar{u}_{\lambda'}(\vec{p}') Q_\nu \gamma^\nu u_\lambda(\vec{p}) = \bar{v}_{\lambda'}(\vec{p}') Q_\nu \gamma^\nu v_\lambda(\vec{p}) = \bar{u}_{\lambda'}(\vec{p}') P_\nu \gamma^\nu v_\lambda(\vec{p}) = \bar{v}_{\lambda'}(\vec{p}') P_\nu \gamma^\nu u_\lambda(\vec{p}) = 0$ , the spinor normalization condition, Eq. (47), and the properties of  $W_\nu(P, Q)$  and  $W_\nu(Q, P)$ , respectively (with  $P = p + p'$ ,  $Q = p - p'$ ). Thus we have proved in the spin-1/2 case that the Fock-space representation of the 4-momentum operator takes on its well known form. With some more effort this can also be verified for rotation and boost generators which finally establishes the equivalence of equal-time quantization and quantization on the hyperboloid  $x \cdot x = \tau^2$  for free spin-1/2 fields.

#### 4 Interacting fields and scattering

After having shown that quantization on the forward hyperboloid and equal-time quantization provide the same Fock-space representation of the Poincaré generators for free fields (if the same set of basis states is taken), we will now investigate the effect of including an interaction term into the Lagrangian density, i.e.  $\mathcal{L}(x) = \mathcal{L}_{\text{free}}(x) + \mathcal{L}_{\text{int}}(x)$ . As long as  $\mathcal{L}_{\text{int}}(x)$  does not contain derivatives of the fields, we infer immediately from Eqs. (36) and (37) that the interacting part of the 4-momentum operator is given by

$$\hat{P}_{\text{int}}^\mu = - \int_{\sigma_\tau} d\sigma_\nu g^{\nu\mu} : \hat{\mathcal{L}}_{\text{int}}(x) : = - \int_{\mathbb{R}^4} 2 d^4 x \delta(x^2 - \tau^2) \theta(x^0) x^\mu : \hat{\mathcal{L}}_{\text{int}}(x) : , \tag{58}$$

where  $\hat{\mathcal{L}}_{\text{int}}(x)$  is short for  $\mathcal{L}_{\text{int}}(\hat{\phi}(x), \partial_\mu \hat{\phi}(x), \dots)$ . This means that all components of the 4-momentum operator become interaction dependent when quantizing on the hyperboloid  $\sigma_\tau$ . Let us next look at the generators of spatial rotations and Lorentz boosts. If we combine them to the antisymmetric tensor  $\hat{M}^{\mu\nu}$  it is not difficult to see that the interaction dependent part of this tensor vanishes:

$$\begin{aligned}
\hat{M}_{\text{int}}^{\mu\nu} &= \int_{\sigma_\tau} d\sigma_\rho \left[ x^\mu \hat{T}_{\text{int}}^{\rho\nu} - x^\nu \hat{T}_{\text{int}}^{\rho\mu} \right] \\
&= - \int_{\mathbb{R}^4} 2 d^4x \delta(x^2 - \tau^2) \theta(x^0) x_\rho [x^\mu g^{\rho\nu} - x^\nu g^{\rho\mu}] : \hat{\mathcal{L}}_{\text{int}}(x) : = 0. \quad (59)
\end{aligned}$$

When quantizing on the hyperboloid  $\sigma_\tau$  the rotation and boost generators are thus not affected by interactions.

For the discussion of scattering in interacting quantum field theories it is most convenient to use the interaction picture. In the interaction picture the evolution of operators is the same as for the free system whereas the interaction determines the evolution of the states. If one considers the evolution of the system generated by the 4-momentum operator, the interaction picture in PFQFT can be cast into a nice covariant form, by splitting the 4-momentum operator into a free and an interacting part,

$$\hat{P}^\mu = \hat{P}_{\text{free}}^\mu + \hat{P}_{\text{int}}^\mu, \quad (60)$$

where each term is obtained by quantizing on the forward hyperboloid. Since all components of the 4-momentum operator are interaction dependent it makes sense to adapt the interaction picture in such a way that it covers evolution into arbitrary space-time directions. Let  $\hat{\mathcal{O}}$  be an operator and  $|\psi\rangle$  be a state specified on the quantization surface  $\sigma_\tau$ . Evolution of the system into the  $x_\mu$ -direction is then described by the set of equations

$$i\partial^\mu \hat{\mathcal{O}}(x) = [\hat{\mathcal{O}}(x), \hat{P}_{\text{free}}^\mu], \quad \hat{\mathcal{O}}(x=0) = \hat{\mathcal{O}}, \quad (61)$$

$$i\partial^\mu |\psi(x)\rangle = \hat{P}_{\text{int}}^\mu(x) |\psi(x)\rangle, \quad |\psi(x=0)\rangle = |\psi\rangle, \quad (62)$$

where

$$\hat{P}_{\text{int}}^\mu(x) := e^{i\hat{P}_{\text{free}} \cdot x} \hat{P}_{\text{int}}^\mu e^{-i\hat{P}_{\text{free}} \cdot x}. \quad (63)$$

Equation (62) is formally solved by introducing an evolution operator  $\hat{U}(y, x)$  such that

$$\hat{U}(y, x) |\psi(x)\rangle = |\psi(y)\rangle. \quad (64)$$

Applying  $\partial^\mu$  to both sides of this equation we infer with the help of Eq. (62) that  $\hat{U}(y, x)$  has to satisfy the differential equation

$$i \frac{\partial}{\partial y_\mu} \hat{U}(y, x) = \hat{P}_{\text{int}}^\mu(y) \hat{U}(y, x), \quad \text{with} \quad \hat{U}(x, x) = \hat{1}. \quad (65)$$

For further purposes it is more convenient to rewrite this initial-value problem for  $\hat{U}(x, x_0)$  as an integral equation:

$$\hat{U}(y, x) = \hat{1} - i \int_{\mathcal{C}(x, y)} dy'_\mu \hat{P}_{\text{int}}^\mu(y') \hat{U}(y', x). \quad (66)$$

The integral runs along an arbitrary smooth path  $\mathcal{C}(x, y)$  joining  $x$  with  $y$ . The formal solution of this integral equation can be written as a path-ordered exponential

$$\hat{U}(y, x) = \mathcal{P} \exp \left( -i \int_{\mathcal{C}(x, y)} dy'_\mu \hat{P}_{\text{int}}^\mu(y') \right). \quad (67)$$

This result may be used to define the scattering operator:

$$\hat{S} = \lim_{x^2, y^2 \rightarrow +\infty} \hat{U}(y, x) \quad \text{such that} \quad x^0 < 0, y^0 > 0, \quad (68)$$

i.e. the limits are taken in such a way that  $y$  stays in the forward and  $x$  in the backward light cone, respectively. Since the path  $\mathcal{C}(x, y)$  can be chosen arbitrarily, we can take a straight line joining  $x$  and  $y$ . The path for the calculation of the scattering operator may thus be parameterized as

$$y'_\mu(s) = a_\mu + s n_\mu \quad (69)$$

with the timelike 4-vector  $n$  given by

$$n = \lim_{x^2, y^2 \rightarrow +\infty} \frac{y - x}{\sqrt{(y - x)^2}}, \quad \text{such that} \quad n \cdot n = 1, \quad (70)$$

and  $a_\mu$  appropriately chosen. With this parameterization the scattering operator becomes a simple  $s$ -ordered exponential (which we indicate by  $\mathcal{S}$  in front of the exponential)

$$\hat{S} = \mathcal{S} \exp \left( -i \int_{-\infty}^{\infty} ds n_\mu \hat{P}_{\text{int}}^\mu(y'(s)) \right). \quad (71)$$

In order to check whether Eq. (71) provides a sensible definition of the scattering operator we first expand the exponential up to leading order in the

interaction:

$$\hat{S} = \hat{1} - i \int_{-\infty}^{\infty} ds n_{\mu} \hat{P}_{\text{int}}^{\mu}(y'(s)) + \dots \quad (72)$$

By means of Eqs. (58) and (63) we get

$$\begin{aligned} \hat{S} &= \hat{1} + i \int_{-\infty}^{\infty} ds n_{\mu} \int_{\mathbb{R}^4} 2 d^4x \delta(x^2 - \tau^2) \theta(x^0) x^{\mu} : \hat{\mathcal{L}}_{\text{int}}(x + y'(s)) : + \dots \\ &= \hat{1} + i \int_{-\infty}^{\infty} ds \int_{\mathbb{R}^3} \frac{d^3x}{x^0} n \cdot x : \hat{\mathcal{L}}_{\text{int}}(x + a + sn) : + \dots \end{aligned} \quad (73)$$

Since  $n$  is a timelike vector it can be written as  $n = \Lambda_v \tilde{n}$  with  $\tilde{n} = (1, 0, 0, 0)$  and  $\Lambda_v$  an appropriate boost. Further, taking into account that  $d^3x/x^0$  is a Lorentz invariant integration measure the integral over the Lagrangian density can be written as

$$\hat{S} = \hat{1} + i \int_{-\infty}^{\infty} ds \int_{\mathbb{R}^3} d^3\tilde{x} : \hat{\mathcal{L}}_{\text{int}}(\Lambda_v(\tilde{x} + \tilde{a} + s\tilde{n})) : + \dots, \quad (74)$$

with  $\tilde{x} = \hat{\Lambda}_v^{-1}x$  and  $\tilde{a} = \hat{\Lambda}_v^{-1}a$ . After a further change of coordinates ( $z = (\sqrt{\tau^2 + \vec{x}^2} + \tilde{a}^0 + s, \vec{x} + \vec{\tilde{a}})$ ,  $d^4z = ds d^3\tilde{x}$ ) we finally obtain

$$\hat{S} = \hat{1} + i \int_{\mathbb{R}^4} d^4z : \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z) : + \dots = \hat{1} + i \int_{\mathbb{R}^4} d^4z' : \hat{\mathcal{L}}_{\text{int}}(z') : + \dots, \quad (75)$$

with  $z' = \Lambda_v z$ . This result, however, is nothing else than the familiar leading-order perturbative expression for the scattering operator.

Beyond leading order in the interaction  $s$ -ordering comes into play. The second-order contribution to the scattering operator is, e.g., given by

$$\begin{aligned} \hat{S}^{(2)} &= (-i)^2 \int_{-\infty}^{\infty} ds_1 n_{\mu} \hat{P}_{\text{int}}^{\mu}(y'(s_1)) \int_{-\infty}^{s_1} ds_2 n_{\nu} \hat{P}_{\text{int}}^{\nu}(y'(s_2)) \\ &= \frac{(-i)^2}{2!} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 n_{\mu} n_{\nu} \mathcal{S} \left[ \hat{P}_{\text{int}}^{\mu}(y'(s_1)) \hat{P}_{\text{int}}^{\nu}(y'(s_2)) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{(-i)^2}{2!} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 \int_{\mathbb{R}^4} 2d^4 x_1 \delta(x_1^2 - \tau^2) \theta(x_1^0) \int_{\mathbb{R}^4} 2d^4 x_2 \delta(x_2^2 - \tau^2) \theta(x_2^0) \\
&\quad \times n_\mu x_1^\mu n_\nu x_2^\nu \left[ \theta(s_1 - s_2) : \hat{\mathcal{L}}_{\text{int}}(x_1 + a + s_1 n) :: \hat{\mathcal{L}}_{\text{int}}(x_2 + a + s_2 n) : \right. \\
&\quad \left. + \theta(s_2 - s_1) : \hat{\mathcal{L}}_{\text{int}}(x_2 + a + s_2 n) :: \hat{\mathcal{L}}_{\text{int}}(x_1 + a + s_1 n) : \right] \\
&= \frac{(-i)^2}{2!} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 \int_{\mathbb{R}^3} d^3 \tilde{x}_1 \int_{\mathbb{R}^3} d^3 \tilde{x}_2 \\
&\quad \times \left[ \theta(s_1 - s_2) : \hat{\mathcal{L}}_{\text{int}}(\Lambda_v(\tilde{x}_1 + \tilde{a} + s_1 \tilde{n})) :: \hat{\mathcal{L}}_{\text{int}}(\Lambda_v(\tilde{x}_2 + \tilde{a} + s_2 \tilde{n})) : \right. \\
&\quad \left. + \theta(s_2 - s_1) : \hat{\mathcal{L}}_{\text{int}}(\Lambda_v(\tilde{x}_2 + \tilde{a} + s_2 \tilde{n})) :: \hat{\mathcal{L}}_{\text{int}}(\Lambda_v(\tilde{x}_1 + \tilde{a} + s_1 \tilde{n})) : \right], \tag{76}
\end{aligned}$$

with  $\tilde{n}$ ,  $\Lambda_v$ ,  $\tilde{a}$  defined as in the leading-order case and  $x_i = \Lambda_v \tilde{x}_i$ . With a further change of coordinates ( $z_i = (\sqrt{\tau^2 + \vec{x}_i^2} + \tilde{a}^0 + s_i, \vec{x}_i + \vec{a})$ ,  $d^4 z_i = ds d^3 \tilde{x}_i$ ,  $i = 1, 2$ ) and the abbreviation  $d_{1,2} = \left( \sqrt{\tau^2 + (\vec{z}_1 - \vec{a})^2} - \sqrt{\tau^2 + (\vec{z}_2 - \vec{a})^2} \right)$  Eq. (76) becomes

$$\begin{aligned}
\hat{S}^{(2)} &= \frac{(-i)^2}{2!} \int_{\mathbb{R}^4} d^4 z_1 \int_{\mathbb{R}^4} d^4 z_2 \left[ \theta(z_1^0 - z_2^0 - d_{1,2}) : \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_1) :: \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_2) : \right. \\
&\quad \left. + \theta(z_2^0 - z_1^0 + d_{1,2}) : \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_2) :: \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_1) : \right] \\
&= \frac{(-i)^2}{2!} \left\{ \int_{\mathbb{R}^4} d^4 z_1 \int_{\mathbb{R}^4} d^4 z_2 \left[ \theta(z_1^0 - z_2^0) : \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_1) :: \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_2) : \right. \right. \\
&\quad \left. \left. + \theta(z_2^0 - z_1^0) : \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_2) :: \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_1) : \right] \right. \\
&\quad \left. + \int_{\mathbb{R}^4} d^4 z_1 \int_{\mathbb{R}^3} d^3 z_2 \int_{z_1^0 - d_{1,2}}^{z_1^0} dz_2^0 : \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_1) :: \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_2) : \right. \\
&\quad \left. + \int_{\mathbb{R}^4} d^4 z_1 \int_{\mathbb{R}^3} d^3 z_2 \int_{z_1^0 - d_{1,2}}^{z_1^0} dz_2^0 : \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_2) :: \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_1) : \right\}. \tag{77}
\end{aligned}$$

If we now concentrate on the last two integrals we observe, that due to the restriction on the  $z_2^0$ -integration  $z_2$  and  $z_1$  are always separated by a spacelike distance. Therefore  $: \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_1) :$  and  $: \hat{\mathcal{L}}_{\text{int}}(\Lambda_v z_2) :$  can be interchanged and the two integrals cancel each other. For the remaining integral we make a final change of variables  $z'_i = \Lambda_v z_i$  and make use of  $\theta(z'_i - z'_j) = \theta(z_i - z_j)$  to obtain

$$\hat{S}^{(2)} = \frac{(-i)^2}{2!} \int_{\mathbb{R}^4} d^4 z'_1 \int_{\mathbb{R}^4} d^4 z'_2 \left[ \theta(z_1^{0'} - z_2^{0'}) : \hat{\mathcal{L}}_{\text{int}}(z'_1) :: \hat{\mathcal{L}}_{\text{int}}(z'_2) : \right. \\ \left. + \theta(z_2^{0'} - z_1^{0'}) : \hat{\mathcal{L}}_{\text{int}}(z'_2) :: \hat{\mathcal{L}}_{\text{int}}(z'_1) : \right]. \quad (78)$$

But this is again nothing else than the second-order contribution for the S-operator in usual time-ordered perturbation theory. The way of reasoning just outlined for the two lowest orders can be generalized to higher orders in the interaction so that Eq. (71) is indeed seen to be equivalent to the well known (instant-form) representation of the S operator as the usual time-ordered exponential [3]. The equivalence of Eq. (71) with the usual perturbative expression for the S-operator tells us also that total 4-momentum conservation is guaranteed by our formulation of scattering, although 3-momentum conservation at the vertices does, in general, not hold within PFQFT. It remains to be seen whether any advantages can be drawn from the manifestly covariant representation of the S-operator, as given in Eq. (71) (or even more general in Eqs. (67) and (68)), to organize perturbative calculations.

## 5 Summary and Outlook

Point-form quantum field theory was first developed in the 1970's by taking the forward hyperboloid  $x_\mu x^\mu = \tau^2$  as the quantization surface. All 4 components of the momentum operator are then dynamic in the sense that they evolve the system away from the quantization surface. The generators of Lorentz transformations, on the other hand, are purely kinematic. It seemed thus to be quite natural to choose a Fock-space basis which is related to the generators of the Lorentz group (in contrast to the more usual momentum basis used in instant- and front-form quantum field theories). The advantage of this “Lorentz basis” is that the quantum numbers which label the basis states are conserved at interaction vertices, whereas 4-momentum is not. In these earlier papers emphasis was put on studying the evolution in  $\tau$ , i.e. the evolution generated by the dilatation operator. However, as was realized very soon, and as is discussed in Sec. 2, the evolution in  $\tau$  together with the Lorentz basis leads to a number of conceptual difficulties which stopped the further development of point-form quantum field theory.

It is, nevertheless, still possible to carry out the Schwinger-Tomonaga program for quantizing on a curved hypersurface like the forward hyperboloid. In this paper we have developed a point-form quantum field theory in a momentum basis. In such a basis overall energy **and** 3-momentum are, in general, not conserved in intermediate states. Neither is the free overall 4-velocity. But overall 4-momentum conservation holds, of course, between the (asymptotic) initial and final states, as our perturbative analysis of the scattering operator has



revealed. This is to be contrasted with point-form relativistic quantum mechanics for finite degree-of-freedom systems, where the overall free 4-velocity is usually chosen to be conserved at each interaction vertex [31]. Then overall energy and momentum conservation is achieved from overall mass conservation. Thus, when interacting mass operators in point form relativistic quantum mechanics are obtained from quantum field theoretic vertices, the requirement of 4-velocity conservation must be added as an explicit requirement [23].

We have shown how to analyze free (spin 0 and 1/2) quantum fields, where the inner product is given by integration over the forward hyperboloid (see Sec. 3). With such an integration surface canonical quantization can be formulated in a manifestly Lorentz covariant way, without making reference to a particular time parameter. When interactions arising from products of local fields are generated, all of the interactions are in the 4-momentum operator, and Lorentz generators are kinematic. A convenient way to express the fact that quantization of a local field theory on the forward hyperboloid provides a representation of the Poincaré algebra are the “point-form” equations,

$$[\hat{P}^\mu, \hat{P}^\nu] = 0 \quad (79)$$

$$\hat{U}_\Lambda \hat{P}^\mu \hat{U}_\Lambda^{-1} = (\Lambda^{-1})^\mu_\nu \hat{P}^\nu, \quad (80)$$

where  $\hat{P}^\mu$  is the total four-momentum operator (including all interactions) and  $\hat{U}_\Lambda$  is the unitary operator implementing the Lorentz transformations. Adopting the Schrödinger picture (indicated by a subscript “S”), these equations lead naturally to a covariant Schrödinger equation,

$$i\partial^\mu |\psi(x)\rangle_S = \hat{P}^\mu |\psi(x)\rangle_S, \quad (81)$$

where  $|\psi(x=0)\rangle_S$  is the state of the system specified on the quantization surface. This covariant Schrödinger equation can be used for the solution of the quantum field theoretic bound-state problem. If the total four-momentum operator is written as a free plus interacting part,  $\hat{P}^\mu = \hat{P}_{\text{free}}^\mu + \hat{P}_{\text{int}}^\mu$ , a generalized interaction representation is easily obtained, which is the starting point for the covariant formulation of scattering given in Sec. 4.

The nice feature that the operator formalism becomes manifestly Lorentz covariant if fields are quantized on the forward hyperboloid is not the only reason to study point-form quantum field theory. Relativistic quantum mechanical models (with a finite number of particles) often rely on field theoretical ideas. Thus it is quite natural to take point-form quantum field theory as a starting point for the construction of effective interactions, currents, etc., which can be applied to point-form quantum mechanics. A further motivation for developing a point form quantum field theory is to analyze gauge theories.

Because Lorentz transformations are kinematic and the theory is manifestly Lorentz covariant, gauge transformations and gauge invariance can be naturally incorporated into the theory. Thus, a promising application will, e.g., be to view quantum chromodynamics as a point form quantum field theory and investigate the nature of gauge fixing and other properties of non-Abelian gauge theories.

## A The distribution $W(P, Q)$

The distribution  $W(P, Q)$  is used repeatedly to perform integrations over the space-time hyperboloid  $x^2 = \tau^2$ . This appendix summarizes the properties of  $W(P, Q)$  needed for the derivations in the various sections. To begin with,  $W(P, Q)$  is defined as

$$W(P, Q) := 2 \int d^4x \delta(x \cdot x - \tau^2) \theta(x_0) x \cdot P e^{-ix \cdot Q}. \quad (\text{A.1})$$

In the relevant cases we have in addition  $P \cdot Q = 0$ ,  $P$  timelike and  $Q$  spacelike. Since  $P$  is timelike, it can be written as

$$P = \hat{B}_c(v) \begin{pmatrix} M \\ \vec{0} \end{pmatrix} \quad \text{with} \quad M^2 = P_\mu P^\mu. \quad (\text{A.2})$$

$\hat{B}_c(v)$  is a rotationless canonical boost with velocity  $v = P/M$ . Its explicit form is [30]

$$\hat{B}_c(v) = \begin{pmatrix} v^0 & \vec{v}^T \\ \vec{v} & \mathbf{1} + \frac{v^0-1}{v^2} \vec{v} \vec{v}^T \end{pmatrix} = \begin{pmatrix} P^0/M & \vec{P}^T/M \\ \vec{P}/M & \mathbf{1} + \frac{P^0/M-1}{\vec{P}^2} \vec{P} \vec{P}^T \end{pmatrix}. \quad (\text{A.3})$$

Since  $Q$  is orthogonal to  $P$  we have

$$Q = \begin{pmatrix} Q^0 \\ \vec{Q} \end{pmatrix} = \hat{B}_c(v) \begin{pmatrix} 0 \\ \vec{q} \end{pmatrix}. \quad (\text{A.4})$$

Inverting Eq. (A.4) gives

$$\vec{q} = \vec{Q} - \frac{\vec{v} \cdot \vec{Q}}{v_0 + v_0^2} \vec{v} = \mathbf{N} \vec{Q}, \quad (\text{A.5})$$

where we have used the orthogonality of  $P$  and  $Q$  to express  $Q^0$  as  $Q^0 = \vec{v} \cdot \vec{Q}/v^0$ .  $\mathbf{N}$  is a  $3 \times 3$ -matrix with elements  $N_{ij} = \delta_{ij} - v_i v_j / (v_0 + v_0^2)$  and determinant  $\det(\mathbf{N}) = 1/v_0$ . By definition  $W(P, Q)$  is Lorentz invariant (cf. Eq. (A.1)) so that it can be easily calculated in the frame where  $P$  has vanishing spacelike components:

$$\begin{aligned} W(P, Q) &= 2 \int d^4x \frac{\delta(x^0 - \sqrt{\tau^2 + \vec{x}^2})}{2x^0} x^0 M e^{i\vec{x} \cdot \vec{q}} = \int d^3x M e^{i\vec{x} \cdot \vec{q}} \\ &= (2\pi)^3 M \delta^3(\vec{q}). \end{aligned} \quad (\text{A.6})$$

Going back to the original frame we finally get

$$\begin{aligned} W(P, Q) &= (2\pi)^3 M \delta^3(\vec{q}) = (2\pi)^3 M \delta^3(\mathbf{N} \vec{Q}) = (2\pi)^3 \frac{M}{\det(\mathbf{N})} \delta^3(\vec{Q}) \\ &= (2\pi)^3 M v_0 \delta^3(\vec{Q}) = (2\pi)^3 P_0 \delta^3(\vec{Q}), \quad P \cdot P > 0 \text{ and } P \cdot Q = 0. \end{aligned} \quad (\text{A.7})$$

If  $P$  and  $Q$  are interchanged  $W(Q, P)$  becomes zero. Proceeding in the same way as before one has

$$\begin{aligned} W(Q, P) &= 2 \int d^4x \frac{\delta(x^0 - \sqrt{\tau^2 + \vec{x}^2})}{2x^0} \vec{x} \cdot \vec{q} e^{-ix^0 M} \\ &= \int d^3x \frac{\vec{x} \cdot \vec{q}}{\sqrt{\tau^2 + \vec{x}^2}} e^{-i\sqrt{\tau^2 + \vec{x}^2} M} = 0, \quad P \cdot P > 0 \text{ and } P \cdot Q = 0, \end{aligned} \quad (\text{A.8})$$

since the integrand is odd in  $\vec{x}$ .

It is also useful to introduce a Lorentz vector

$$W_\mu(P, Q) := \frac{P_\mu}{P \cdot P} W(P, Q) + \frac{Q_\mu}{Q \cdot Q} W(Q, Q), \quad (\text{A.9})$$

with  $P \cdot P > 0$ ,  $P \cdot Q = 0$ , and  $W(.,.)$  defined according to Eq. (A.1). Further, if Eq.(A.1) is differentiated with respect to  $P^\mu$ , one encounters the distribution

$$W_\mu^\tau(Q) := 2 \int d^4x \delta(x \cdot x - \tau^2) \theta(x_0) x_\mu e^{-ix \cdot Q}. \quad (\text{A.10})$$

It is then straightforward to show that  $W_\mu^\tau = W_\mu(P, Q)$ . We introduce 2 additional spacelike 4-vectors  $R$  and  $S$  which, together with  $P$  and  $Q$ , form

an orthogonal basis of Minkowski space. If one now represents  $x_\mu$  in terms of these basis vectors, the right-hand side of Eq. (A.10) becomes

$$\frac{P_\mu}{P \cdot P} W(P, Q) + \frac{Q_\mu}{Q \cdot Q} W(Q, Q) + \frac{R_\mu}{R \cdot R} W(R, Q) + \frac{S_\mu}{S \cdot S} W(S, Q). \quad (\text{A.11})$$

When calculating  $W(R, Q)$  and  $W(S, Q)$  we can choose our coordinate system for the integration variables such that the spatial coordinate axes coincide with  $Q$ ,  $R$ , and  $S$ , respectively. For this choice of coordinates it is immediately obvious that the integrand of  $W(R, Q)$  is odd in the  $R$ -direction and thus  $W(R, Q) = 0$ . An analogous reasoning holds for  $W(S, Q)$ , which proves Eq. (A.10). The specific form of the Lorentz scalar  $W(Q, Q)$  is not as simple as that of  $W(P, Q)$ , but it is not required in our derivations.

By a similar reasoning we see that

$$W_\mu(Q, P) := \frac{P_\mu}{P \cdot P} W(P, P) = 2 \int d^4x \delta(x \cdot x - \tau^2) \theta(x_0) x_\mu e^{-ix \cdot P}. \quad (\text{A.12})$$

Actually  $W_\mu(Q, P)$  does not depend on  $Q$ . We have only kept  $Q$  in its argument to better exhibit symmetry properties under exchange of  $P$  and  $Q$  in intermediate steps of our calculations.

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## References

- [1] S. Tomonaga, Prog. Theor. Phys. 1 (1946) 27.
- [2] J. Schwinger, Phys. Rev. 74 (1948) 1439.
- [3] S. S. Schweber, An Introduction to Relativistic Quantum Field Theory, Row, Peterson and Company, Elmsford, New York, 1961.
- [4] P. A. M. Dirac, Rev. Mod. Phys. 21 (1949) 329.
- [5] H. Leutwyler, J. Stern, Ann. Phys. (N.Y.) 112 (1978) 94.
- [6] F. Coester, Prog. Part. Nucl. Phys. 29 (1992) 1.

- [7] S. J. Brodsky, H.-C. Pauli, Phys. Rep. 301 (1998) 299.
- [8] S. Fubini, A. J. Hanson, R. Jackiw, Phys. Rev. D 7 (1973) 1732.
- [9] C. M. Sommerfield, Ann. Phys. (N.Y.) 84 (1974) 285.
- [10] D. Gromes, H. J. Rothe, B. Stech, Nucl. Phys. B 75 (1974) 313.
- [11] A. di Sessa, Phys. Rev. D 9 (1974) 2926.
- [12] A. di Sessa, J. Math. Phys. 15 (1974) 1892.
- [13] PFQFT can, of course, be considered as a special case of quantization of field theories in curved space-time, which is an extensively studied topic. But global Poincaré invariance and hence the usual concept of a particle loses its meaning within such a general framework so that the particle aspects of PFQFT are not immediately obvious. For a comprehensive discussion of field quantization in curved space-time see, e.g.,  
R. M. Wald, Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics, The University of Chicago Press, Chicago, London, 1994.
- [14] F. Lev, Ann. Phys. (N.Y.) 237 (1995) 355.
- [15] W. H. Klink, Nucl. Phys. A 716 (2003) 136.
- [16] T. Melde, L. Canton, W. Plessas, R. F. Wagenbrunn, nucl-th/0612013 (2006).
- [17] L. Y. Glozman, W. Plessas, K. Varga, R. F. Wagenbrunn, Phys. Rev. D 58 (1998) 094030.
- [18] K. Glantschnig, R. Kainhofer, W. Plessas, B. Sengl, R. F. Wagenbrunn, Eur. Phys. J. A 23 (2005) 507.
- [19] S. Boffi, L. Y. Glozman, W. Klink, W. Plessas, M. Radici, R.F. Wagenbrunn, Eur. Phys. J. A 14 (2002) 17.
- [20] K. Berger, R. F. Wagenbrunn, W. Plessas, Phys. Rev. D 70 (2004) 094027.
- [21] T. Melde, W. Plessas, R. F. Wagenbrunn, Phys. Rev. C72 (2005) 015207; Erratum-ibid. C 74 (2006) 069901.
- [22] T. Melde, W. Plessas, B. Sengl, nucl-th/0612053 (2006).
- [23] W. H. Klink, Nucl. Phys. A 716 (2003) 123.
- [24] A. Krassnigg, W. Schweiger, W. H. Klink, Phys. Rev. C 67 (2003) 064003.
- [25] A. Krassnigg, Phys. Rev. C 72 (2005) 028201.
- [26] W. W. MacDowell, R. Roskies, J. Math. Phys. 13 (1972) 1585.
- [27] S. Zelzer, Point-Form Quantum Field Theory, diploma thesis, University Graz (2005).
- [28] C. Itzykson, J. Math. Phys. 10 (1969) 1109.

- [29] S. V. Ketov, Conformal Field Theory, World Scientific, Singapore, 1995.
- [30] W. H. Klink, Phys. Rev. C 58 (1998) 3617.
- [31] B. Bakamjian, L. H. Thomas, Phys. Rev. 92 (1953) 1300.